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Hawking Radiation of Rarita–Schwinger Fields of a Stationary Charged Black Hole

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Hawking radiation of the Rarita–Schwinger fields in a stationary charged black hole is studied exactly in region near the event horizon by using the Newman–Penrose formalism and the tortoise coordinate. The result shows that the temperature of the thermal radiation spectrum of Rarita–Schwinger fields is exactly the same as that of the scalar, Dirac, and electromagnetic fields.

KEY WORDS: Rarita–Schwinger field; Hawking radiation; Newman–Penrose formalism.

Since the quantum thermal radiation of black hole was found by Hawking (1975) in 1975, much attention has been focused on the Hawking radiation. The thermal effect due to the Klein–Gordon scalar field in the Kerr–Newman black hole, the NUT–Kerr–Newman black hole, Vaidya–Schwarzschild–de-Sitter, and the radiating rotating charged black hole is studied by Damour and Ruffini (1976), Ahmed (1987), Dai *et al.* (1993), and Jing and Wang (1997). The quantum thermal radiation of the Dirac field in the near extreme Kerr black hole was investigated by Liu and Xu (1980), the near extreme Kerr–Newman black hole by Zhao and Gui (1983), and the Kerr–Newman–Kasuya by Ahmed and Mondal (1993). Jing (2002) recently studied the quantum thermal effect arising from the electromagnetic fields in the general Kerr–Newman black hole and found that the thermal radiation spectrum due to the photons is same as that of the Klein–Gordon scalar particles.

We all know that Rarita–Schwinger fields (Aichelburg and Güven, 1981; Ferrara and van Nieuwenhuizen, 1976; Güven, 1981; Silva-Ortigoza, 1996, 1997; Torres del Castillo, 1989; van Nieuwenhuizen, 1981a) connect with supergravity theory and supergravity has attractive quantum properties (van Nieuwenhuizen, 1981). It is therefore of considerable interest to ask whether the Rarita–Schwinger fields may bring new features into black holes. The purpose of this paper is to

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study the quantum thermal effect of the Rarita–Schwinger fields in the stationary charged black hole.

The supergravity field equations reduce to the equations of general relativity when the Rarita–Schwinger fields vanish. If the supergravity field equations (Ferrara and van Nieuwenhuizen, 1976; Townsend, 1977; van Nieuwenhuizen, 1981b) are linearized with respect to the spin-$\frac{3}{2}$ fields about a solution with vanishing spin-$\frac{3}{2}$ fields, a consistent set of equations for the Rarita–Schwinger fields on a background spacetime is given by (Aichelburg and Güven, 1981)

$$\Phi_{ABC'D'} = 2\phi_{AB}\bar{\phi}_{C'D'}, \tag{1}$$

$$\Lambda = \frac{R}{24} = 0, \tag{2}$$

$$\nabla_{AB'}\psi^{jA}_{CD'} = \nabla_{CD'}\psi^{jA}_{AB'} - i\sqrt{2}\epsilon^{jk}\phi^{A}_{C}\bar{\psi}^{k}_{D'B'A}, \tag{3}$$

where $j, k = 1, 2$, $\Phi_{ABC'D'}$ is the trace-free part of the Ricci tensor, ϕ_{AB} is the electromagnetic spinor, and ϵ^{jk} is the usual Levi–Civita symbol. Equation (1) is the usual Einstein–Maxwell equation and Eq. (3) can be rewritten as

$$H^{j}_{ABC} = H^{j}_{(ABC)}, \qquad H^{j}_{AB'C'} = 0, \tag{4}$$

with

$$\begin{aligned} H^{j}_{ABC} &\equiv \nabla^{R'}_{(B\psi j|A|C)R'} - i\sqrt{2}\epsilon^{jk}\phi_{A(B\psi kR'|R'|C)}, \\ H^{jA}_{B'C'} &\equiv \nabla^{D}_{(B'\psi jA|D|C')} - i\sqrt{2}\epsilon^{jk}\phi^{AR}\psi^{k}_{(B'C')R}, \end{aligned} \tag{5}$$

where the parentheses denote symmetrization on the indices enclosed and the indices between bars are excluded from the symmetrization. These equations are consistent if the background gravitational and electromagnetic fields satisfy the coupled Einstein–Maxwell equations

$$\nabla^{A}_{C'}\phi_{AB} = 0, \tag{6}$$

and then they are invariant under the supersymmetry transformations

$$\psi^{j}_{ACD'} \to \psi^{j}_{ACD'} + \nabla_{CD'}\alpha^{j}_{A} - i\sqrt{2}\epsilon^{jk}\phi_{AC}\bar{\alpha}^{k}_{D'}, \tag{7}$$

where α^{j}_{A} is a pair of arbitrary spinor fields and $\alpha^{k}_{C'} = \overline{\alpha^{k}_{C}}$. By using Eqs. (3) and (5) and the Ricci identities, one finds that H^{j}_{ABC} satisfy (Torres del Castillo, 1989)

$$\nabla^{AR'}H^{j}_{ABC} = \Psi_{ABC}{}^{D}\psi^{jA}{}_{D}{}^{R'} + i\sqrt{2}\epsilon^{jk}\psi^{KS'R'A}\nabla_{BS'}\phi_{AC}, \tag{8}$$

where Ψ_{ABCD} is the Weyl spinor. This spinor dyad defines at each point a null tetrad $(l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$. For the case that the background field is algebraically general and their principal null directions are aligned with those of the Weyl spinor, in a spin frame o^A, ℓ^A, such that ϕ_1 and ψ_2 are the only nonvanishing components of

ϕ_{AB} and Ψ_{ABCD}, respectively. By applying Eqs. (3), (5) and the Bianchi and Ricci identities, we know that the decoupled equations are given by (Torres del Castillo, 1989)

$$\begin{aligned}
&[(D - 2\varepsilon + \bar{\varepsilon} - 3\rho - \bar{\rho})(\Delta - 3\gamma + \mu) \\
&\quad - (\delta - 2\beta - \bar{\alpha} - 3\tau + \bar{\pi})(\bar{\delta} - 3\alpha + \pi) - \Psi_2]H_0^j = 0, \\
&[(\Delta + 2\gamma - \bar{\gamma} + 3\mu + \bar{\mu})(D + 3\varepsilon - \rho) \\
&\quad - (\bar{\delta} + 2\alpha + \bar{\beta} + 3\pi - \bar{\tau})(\delta + 3\beta - \tau) - \Psi_2]H_3^j = 0,
\end{aligned} \tag{9}$$

where $H_0^j \equiv H_{ABC}^j o^A o^B o^C$ and $H_3^j \equiv H_{ABC}^j \ell^A \ell^B \ell^C$.

In Boyer–Lindquist coordinates the Kerr–Newman black hole can be specified by the null tetrad

$$\begin{aligned}
l_\mu &= \frac{1}{\Delta}(\Delta, -\Sigma, 0, -a\Delta \sin^2\theta), \\
n_\mu &= \frac{1}{2\Sigma}(\Delta, \Sigma, 0, -a\Delta \sin^2\theta), \\
m_\mu &= -\frac{\bar{\rho}}{\sqrt{2}}(ia \sin\theta, 0, -\Sigma, -i(r^2 + a^2) \sin\theta), \\
\bar{m}_\mu &= -\frac{\rho}{\sqrt{2}}(-ia \sin\theta, 0, -\Sigma, i(r^2 + a^2) \sin\theta),
\end{aligned} \tag{10}$$

with

$$\Sigma = r^2 + a^2 \cos^2\theta, \qquad \Delta = (r - r_+)(r - r_-), \qquad r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}, \tag{11}$$

where r_+, M, a, and Q represent the radius of the event horizon, the mass, the angular momentum per unit mass, and the electric charge of the black hole, respectively. The only nonvanishing spin-coefficients, Weyl spinor and electromagnetic field components are (Chandrasekhar, 1992)

$$\rho = -\frac{1}{r - ia\cos\theta}, \quad \beta = -\frac{\bar{\rho}\cot\theta}{2\sqrt{2}}, \quad \pi = \frac{ia\rho^2 \sin\theta}{\sqrt{2}}, \quad \tau = -\frac{ia\rho\bar{\rho}\sin\theta}{\sqrt{2}},$$

$$\mu = \frac{\rho^2\bar{\rho}\Delta}{2}, \quad \gamma = \mu + \frac{\rho\bar{\rho}(r - M)}{2}, \quad \alpha = \pi - \bar{\beta}, \tag{12}$$

$$\Psi_2 = \rho^3\bar{\rho}\left(\frac{M}{\bar{\rho}} + Q^2\right), \qquad \phi_1 = \frac{Q}{2}(\rho\bar{\rho}), \tag{13}$$

Equations (10), (12), (13), and (9) show that H_0^j and H_3^j admit separable solutions

$$H_0^j = a^j e^{i(Et+m\varphi)} R_{+3/2}(r)\Theta_{+3/2}(\theta), \; H_3^j = b^j \rho^3 e^{i(Et+m\varphi)} R_{-3/2}(r)\Theta_{-3/2}(\theta), \tag{14}$$

where a^j and b^j are arbitrary constants and the functions $R_{\pm 3/2}(r)$ and $\Theta_{\pm 3/2}(\theta)$ satisfy (Torres del Castillo, 1989)

$$
\begin{aligned}
&[\Delta \mathcal{D}_{-1/2}\mathcal{D}_0^\dagger + 4iEr]\Delta^{3/2}R_{+3/2}(r) = \lambda^2\Delta^{3/2}R_{+3/2}(r),\\
&[\Delta \mathcal{D}_{-1/2}^\dagger\mathcal{D}_0 - 4iEr]R_{-3/2}(r) = \lambda^2 R_{-3/2}(r),\\
&[\mathcal{L}_{-1/2}^\dagger\mathcal{L}_{3/2} + 4Ea\cos\theta]\Theta_{+3/2}(\theta) = -\lambda^2\Theta_{+3/2}(\theta)\\
&[\mathcal{L}_{-1/2}\mathcal{L}_{3/2}^\dagger - 4Ea\cos\theta]\Theta_{-3/2}(\theta) = -\lambda^2\Theta_{-3/2}(\theta)
\end{aligned}
\tag{15}
$$

with

$$
\begin{aligned}
&\mathcal{D}_n = \frac{\partial}{\partial r} + \frac{iK_1}{\Delta} + 2n\frac{r-M}{\Delta}, \qquad \mathcal{D}_n^\dagger = \frac{\partial}{\partial r} - \frac{iK_1}{\Delta} + 2n\frac{r-M}{\Delta},\\
&\mathcal{L}_n = \frac{\partial}{\partial\theta} + K_2 + n\cot\theta, \qquad \mathcal{L}_n^\dagger = \frac{\partial}{\partial\theta} - K_2 + n\cot\theta,
\end{aligned}
\tag{16}
$$

where $K_1 = (r^2+a^2)E - ma$ and $K_2 = aE\sin\theta - \frac{m}{\sin\theta}$, E and m are the energy and angular momentum of the Rarita–Schwinger particle, respectively. The decoupled equations (15) can then be explicitly expressed as

$$
\begin{aligned}
&\Delta\frac{d^2R_s}{dr^2} + 5(r-M)\frac{dR_s}{dr}\\
&\quad + \left[2s + 4isrE + \frac{K_1^2 - 2isK_1(r-M)}{\Delta} - \lambda^2\right]R_s = 0, \quad \left(s = +\frac{3}{2}\right),\\
&\Delta\frac{d^2R_s}{dr^2} - (r-M)\frac{dR_s}{dr}\\
&\quad + \left[4isrE + \frac{K_1^2}{\Delta} - \frac{2isK_1(r-M)}{\Delta} - \lambda^2\right]R_s = 0, \quad \left(s = -\frac{3}{2}\right),
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&\frac{d^2\Theta_s}{d\theta^2} + \cot\theta\frac{d\Theta_s}{d\theta} + \left[2maE - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} + 2asE\cos\theta\right.\\
&\quad \left. + \frac{2sm\cos\theta}{\sin^2\theta} - s - s^2\cot^2\theta + \lambda^2\right]\Theta_s = 0, \quad \left(s = +\frac{3}{2}\right)
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\frac{d^2\Theta_s}{d\theta^2} + \cot\theta\frac{d\Theta_s}{d\theta} + \left[2maE - a^2E^2\sin^2\theta - \frac{m^2}{\sin^2\theta} + 2asE\cos\theta\right.\\
&\quad \left. + \frac{2sm\cos\theta}{\sin^2\theta} + s - s^2\cot^2\theta + \lambda^2\right]\Theta_s = 0, \quad \left(s = -\frac{3}{2}\right).
\end{aligned}
\tag{19}
$$

We now introduce the tortoise coordinate in the stationary charged black hole as

$$r_* = r + \frac{1}{2\kappa} \ln(r - r_+) \tag{20}$$

where κ is the surface gravity of the black hole. In the new coordinate Eqs. (17) and (18) can be cast into

$$\frac{d^2 R_s}{dr_*^2} + 2s\kappa \frac{dR_s}{dr_*} + [(\omega - \Omega_H m)^2 - 2isk(\omega - \Omega_H m)]R_s = 0, \quad \left(s = \pm\frac{3}{2}\right), \tag{21}$$

where $\Omega_H = \frac{a}{r_+^2 + a^2}$ is the black hole's angular. Solving Eq. (21) exactly we find that both the radial functions for the $s = \pm\frac{3}{2}$ particles are

$$\begin{aligned} R_s(r_*) &= N_\omega e^{\pm i k_{r*} r*}, \\ k_{r*} &= (\omega - \Omega_H m), \quad \left(\text{for } s = \pm\frac{3}{2}\right), \end{aligned} \tag{22}$$

where N_ω is a integral constant. From above discussion we find that the two linearly independent radial solutions for the Rarita–Schwinger fields can be expressed as

$$\begin{aligned} \Phi^{\text{in}}(v, \hat{r}) &= N_{\text{in}}\, e^{-i\omega v}, \\ \Phi^{\text{out}}(v, \hat{r}) &= N_{\text{out}}\, e^{-i\omega v}\, e^{2i\omega\hat{r}}, \end{aligned} \tag{23}$$

where $v = t + \hat{r} = t + \frac{1}{\omega}(\omega - \Omega_H m)r_*$ is an advanced Eddington–Finkelstein coordinate, $\Phi^{\text{in}}(v, \hat{r})$ represents an incoming wave and is an analytic function on the event horizon; $\Phi^{\text{out}}(v, \hat{r})$, however, represents an outgoing wave and has a logarithmic singularity on the horizon. Near the event horizon, the coordinate $\hat{r}$ tends to

$$\hat{r} \sim \frac{1}{2\kappa} \ln(r - r_+). \tag{24}$$

Thus, we obtain

$$\Phi^{\text{out}}(v, \hat{r}) = N_{\text{out}}\, e^{-i\omega v}(r - r_+)^{i\omega/\kappa}. \tag{25}$$

Since on the horizon the outgoing wave function is not analytic we cannot be extended straightforwardly to the region inside. It must be continued analytically in the complex plane by going around the event horizon. Then we get

$$\Phi^{\text{out}}(v, \hat{r}) = N_{\text{out}}\, e^{-i\omega v}(r_+ - r)^{i\omega/\kappa}\, e^{\pi\omega/\kappa}. \tag{26}$$

The outgoing wave function for both inside and outside region can be written as

$$\begin{aligned} \Phi^{\text{out}}(v, \hat{r}) = N_{\text{out}}\{&y(r - r_+)\, e^{-i\omega v}(r - r_+)\, e^{i\omega/\kappa} \\ &+ y(r_+ - r)\, e^{-i\omega v}(r_+ - r)^{i\omega/\kappa} e^{\pi\omega/\kappa}\}, \end{aligned} \tag{27}$$

where $y(x)$ is a step function

$$y(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{28}$$

By using the suggestion of Damour–Ruffini (1976), and the normalization condition, we get

$$N_{\text{out}}^2 = \frac{\Gamma_\omega}{\exp(2\pi\omega/\kappa) + 1} = \frac{\Gamma_\omega}{\exp[2\pi\omega/(k_{\text{B}}T)] + 1}, \tag{29}$$

with the temperature

$$T = \frac{\kappa}{2\pi k_{\text{B}}}, \tag{30}$$

where Γ_ω is the transmission coefficient caused by the potential barrier in the exterior gravitational field, k_{B} the Boltzmann constant. The formulae (29) and (30) are the main result demonstrating the emission of a thermal spectrum of the Rarita–Schwinger fields in the Kerr–Newman black hole.

To summary, the Rarita–Schwinger fields in the Kerr–Newman black hole is first expressed as the Newman–Penrose formalism and the decouple equations are obtained. By solving the equations exactly in region near the event horizon, the Hawking radiation of the Kerr–Newman black hole are obtained. We find that the quantum thermal radiation spectrum due to the Rarita–Schwinger fields in the Kerr–Newman black hole does not depend on the spins of the particles, and the temperature is exactly same as that arising from the scalar, Dirac, and electromagnetic fields.

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